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Unbounded Divergence of Simple Quadrature Formulas

IOAN MUNTEAN

Faculty of Mathematics,
University Babeş-Bolyai, 3400 Cluj-Napoca, Romania

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Given a sequence of real or complex coefficients c_i and a sequence of distinct nodes t_i in a compact interval T, we prove the divergence and the unbounded divergence on superdense sets in the space $C(T)$ of the simple quadrature formulas

$$\int_T x(t)\,du(t) = Q_n(x) + R_n(x)$$

and

$$\int_T w(t)\,x(t)\,dt = Q_n(x) + R_n(x),$$

where

$$Q_n(x) = \sum_{i=1}^{m_n} c_i x(t_i), \qquad x \in C(T).$$

The divergence (not certainly unbounded) for at most one continuous function of the first simple quadrature formula, with $m_n = n$ and $u(t) = t$, was established by P. J. Davis in 1953.

1. Condensation of Singularities and Unbounded Divergence of Quadrature Formulas

Let X be a normed space over K, where K is the field R of real numbers or the field C of complex numbers. Denote by X^* the dual of X, i.e., the Banach space of all linear and continuous functionals $x^*: X \to K$, endowed with the norm $\|x^*\| = \sup\{|x^*(x)| : x \in X, \|x\| \leqslant 1\}$, $x^* \in X^*$. A subset Y of a topological space Z is said to be *superdense* in Z if Y is an uncountably infinite dense G_δ-set in Z [1, p. 139]. The following principle of condensation of singularities is well-known:

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1.1. THEOREM [1, Theorem 3.1 or 5.4]. *If X is a Banach space and $(x_n^*)_{n\in N}$ is a sequence of elements in X^* such that* $\sup\{\|x_n^*\|: n\in N\}=\infty$, *then the set* $\{x\in X: \sup\{|x_n^*(x)|: n\in N\}=\infty\}$ *is superdense in X.*

Consider a strictly increasing sequence $(m_n)_{n\in N}$ of positive integers, a matrix of coefficients $c_n^1, ..., c_n^{m_n}$ in K, a matrix of distinct nodes $t_n^1, ..., t_n^{m_n}$ in a compact interval $T=[a, b]$, with $a<b$, and a function of bounded variation $u: T\to K$. Denote by $C(T)$ the Banach space of all continuous functions $x: T\to K$ endowed with the norm $\|x\|=\sup\{|x(t)|: t\in T\}$. We say that the quadrature formula

$$\int_T x(t)\,du(t)=Q_n(x)+R_n(x), \qquad x\in C(T), \tag{1}$$

associated with the above data, where

$$Q_n(x)=\sum_{i=1}^{m_n} c_n^i x(t_n^i), \tag{2}$$

is *convergent* on a subset Y of $C(T)$ if for each $y\in Y$ one has $R_n(y)\to 0$ whenever $n\to\infty$. The formula (1) is said to be *unboundedly divergent* on a subset Y of $C(T)$ if $\sup\{|Q_n(y)|: n\in N\}=\infty$ for each $y\in Y$. In what follows we need the next classical characterization of convergent quadrature formulas:

1.2. THEOREM [7, p. 267]. *The quadrature formula* (1) *converges on $C(T)$ if and only if*

(i) *the sequence* $(\sum_{i=1}^{m_n}|c_n^i|)_{n\in N}$ *is bounded;*

(ii) *the formula* (1) *converges on the set P of the restrictions to T of all polynomial functions.*

Defining the linear and continuous functionals $x_n^*: C(T)\to K$ by $x_n^*(x)=Q_n(x)$, $x\in C(T)$, we obtain

$$\|x_n^*\|=\sum_{i=1}^{m_n}|c_n^i|, \tag{3}$$

so that, in virtue of Theorem 1.1, we arrive at the following divergence theorem for quadrature formulas:

1.3. THEOREM. *If*

$$\sup\left\{\sum_{i=1}^{m_n}|c_n^i|: n\in N\right\}=\infty, \tag{4}$$

then the set

$$S=\{x\in C(T):\sup\{|Q_n(x)|:n\in N\}=\infty\}$$

is superdense in $C(T)$; *i.e.*, (1) *is unboundedly divergent on* S.

Given a sequence $(c_i)_{i\in N}$ of coefficients in K and a sequence $(t_i)_{i\in N}$ of distinct nodes in T, we consider the associated quadrature formula

$$\int_T x(t)\,du(t)=Q_n(x)+R_n(x), \qquad x\in C(T), \tag{5}$$

where the corresponding sum $Q_n(x)$ in (2) is defined here by

$$Q_n(x)=\sum_{i=1}^{m_n} c_i x(t_i). \tag{6}$$

We remark that the formulas (5), (6) have the form (1), (2) with $c_n^i=c_i$ and $t_n^i=t_i$, $i\in\{1,\dots,m_n\}$, $n\in N$. When $m_n=n$ and $u(t)=t$, the formula (5) was first introduced by P. J. Davis [2] under the name of *simple quadrature formula.* Davis proved the following divergence theorem for simple quadrature formulas:

1.4. THEOREM [3, pp. 357–358]. *If* $m_n=n$ *and* $u(t)=t$, $t\in T$, *then the quadrature formula* (5) *is divergent*; *in other words, there exists a function* $x_0\in C(T)$ *such that* $R_n(x_0)\not\to 0$ *as* $n\to\infty$.

In this paper we show that Theorem 1.4 remains valid in the case of simple quadrature formulas having the general form (5) (Theorem 2.1). Moreover, we prove the unbounded divergence of (5) on superdense subsets of $C(T)$ (Theorem 3.1), which represents a new result even for the Davis' simple quadrature formula. As an application of these results, we derive the divergence and the unbounded divergence on superdense sets of simple quadrature formulas with a weight (Remark 2.3 and Theorem 3.2).

2. DIVERGENCE OF SIMPLE QUADRATURE FORMULAS

2.1. THEOREM. *If the nodes* t_i, $i\in N$, *are distinct and* $u:T\to K$ *is a non-constant continuous function of bounded variation, then the simple quadrature formula* (5) *is divergent.*

Proof. Assume the contrary, i.e.,

$$\int_T x(t)\,du(t)=\sum_{i=1}^{\infty} c_i x(t_i) \qquad \text{for each} \quad x\in C(T).$$

Then Theorem 1.2 implies

$$\sum_{i=1}^{\infty} |c_i| < \infty. \tag{7}$$

Let us remark that not all c_i in (5), (6) are zero. Indeed, since u is non-constant, there exist p and q in T such that $a \leqslant p < q \leqslant b$ and $u(p) \neq u(q)$. Let $a < p$ and $q < b$. For each positive $\varepsilon < \min\{p-a, b-q\}$ we define the continuous function $x_\varepsilon: T \to R$ by

$$x_\varepsilon(t) = \begin{cases} 1, & \text{if } t \in [p, q], \\ 0, & \text{if } t \in [a, p-\varepsilon] \cup [q+\varepsilon, b], \\ (t-p+\varepsilon)/\varepsilon, & \text{if } t \in]p-\varepsilon, p[, \\ (-t+q+\varepsilon)/\varepsilon, & \text{if } t \in]q, q+\varepsilon[, \end{cases}$$

and observe that $\|x_\varepsilon\| = 1$ and

$$\int_T x_\varepsilon(t)\, du(t) = \int_{p-\varepsilon}^{p} x_\varepsilon(t)\, du(t) + u(q) - u(p) + \int_q^{q+\varepsilon} x_\varepsilon(t)\, du(t). \tag{8}$$

The continuity of the variation function $t \mapsto \bigvee_q^t (u)$ at the point $t = q$ (see [6, pp. 243–244]) implies

$$\left| \int_q^{q+\varepsilon} x_\varepsilon(t)\, du(t) \right| \leqslant \|x_\varepsilon\| \bigvee_q^{q+\varepsilon} (u) \to 0 \qquad \text{as} \qquad \varepsilon \to 0$$

and

$$\int_{p-\varepsilon}^{p} x_\varepsilon(t)\, du(t) \to 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

Now, if all c_i would vanish, take in (8) a sufficiently small $\varepsilon > 0$, make $n \to \infty$ in the formula (5) written for $x = x_\varepsilon$, and arrive at a contradiction.

A similar argument applies when $a = p$ or $q = b$.

Consequently, there must exist a k in N such that

$$\alpha = \tfrac{1}{2} |c_k| > 0. \tag{9}$$

By (7) there is a j in N such that $m_j > k$ and

$$\sum_{i=m_j+1}^{\infty} |c_i| < \alpha. \tag{10}$$

Because the nodes t_i are distinct, we have

$$\beta = \min\{|t_k - t_i| : i \in \{1, ..., m_j\}\setminus\{k\}\} > 0.$$

Suppose first that t_k satisfies $a < t_k < b$ and denote

$$\gamma = \min\{\alpha, \beta, t_k - a, b - t_k\} > 0.$$

Using again the continuity of the function $t \mapsto \bigvee_a^t (u)$ this time at the point $t = t_k$, there exists a $\delta > 0$ with $\delta \leqslant \gamma$ such that

$$\bigvee_{t_k-\delta}^{t_k+\delta} (u) \leqslant \gamma. \tag{11}$$

Now, define the continuous function $x_\delta : T \to R$ by

$$x_\delta(t) = \begin{cases} (t - t_k + \delta)/\delta, & \text{if} \quad t \in \,]t_k - \delta, t_k[, \\ (-t + t_k + \delta)/\delta, & \text{if} \quad t \in [t_k, t_k + \delta[, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Since $\|x_\delta\| = 1$ and $|t_k - t_i| \geqslant \delta$ for all i in $\{1, ..., m_j\}\setminus\{k\}$, the relations (11), (12), (5), (6), (9) and (10) lead to the contradiction

$$\begin{aligned} \alpha \geqslant \gamma \geqslant \|x_\delta\| \cdot \bigvee_{t_k-\delta}^{t_k+\delta} (u) &\geqslant \left| \int_{t_k-\delta}^{t_k+\delta} x_\delta(t)\, du(t) \right| \\ = \left| \int_T x_\delta(t)\, du(t) \right| &= \left| c_k x_\delta(t_k) + \sum_{i=m_j+1}^{\infty} c_i x_\delta(t_i) \right| \\ \geqslant |c_k| - \sum_{i=m_j+1}^{\infty} |c_i| &> 2\alpha - \alpha = \alpha. \end{aligned}$$

When $t_k = a$ or b we use a similar argument. This completes the proof of our theorem. ∎

2.2. *Remarks.* (i) The continuity hypothesis of the function u is not directly used in the proof of Theorem 2.1. We use only the apparently weaker hypothesis concerning the continuity of the variation function $t \mapsto \bigvee_a^t (u)$, $t \in T$. In fact, the continuity of the last function at a point t in T implies the continuity of the given function u at the same point since

$$|u(s) - u(t)| \leqslant \left| \bigvee_a^s (u) - \bigvee_a^t (u) \right|, \qquad s, t \in T.$$

(See also [4, Corollary 1.1]).

(ii) As the next example shows, the continuity of the function u in Theorem 2.1 cannot be dropped. Let $(t_i)_{i \in N}$ be any sequence of distinct nodes in the open interval $]0, 1[$, $(c_i)_{i \in N}$ be any sequence of positive numbers with $\sum_{i=1}^{\infty} c_i < \infty$, and $u, v: R \to R$ be two functions defined by $v(t) = 0$ if $t \leqslant 0$, $v(t) = 1$ if $t > 0$, and $u(t) = \sum_{i=1}^{\infty} c_i v(t - t_i)$. (Note that u is discontinuous at each point t_i.) It can be shown that the corresponding simple quadrature formula (5), (6) with $m_n = n$ is convergent on the space $C[0, 1]$ (see [8, p. 126]).

Let $w: T \to R$ be a weight function, i.e., a nonnegative Lebesgue integrable function on T with

$$\int_T w(t)\, dt > 0. \tag{13}$$

In the same manner as in Section 1 we define the notions of convergence and unbounded divergence of the quadrature formula with the weight w,

$$\int_T w(t)\, x(t)\, dt = Q_n(x) + R_n(x), \qquad x \in C(T), \tag{14}$$

where $Q_n(x)$ is given by (6).

2.3. REMARK. *If the nodes t_i, $i \in N$, are distinct, then the simple quadrature formula* (14) *with the weight w is divergent.*

Proof. As is well-known, the function $u: T \to R$ defined by

$$u(t) = \int_a^t w(s)\, ds$$

is absolutely continuous; hence u has bounded variation on T, possesses a finite derivative a.e. on T, and satisfies $u'(t) = w(t)$ a.e. $t \in T$ (see [6, pp. 271–274]). Therefore, the Riemann–Stieltjes integral $\int_T x(t)\, du(t)$, $x \in C(T)$, exists and is equal to the Lebesgue integral $\int_T x(t)\, u'(t)\, dt$ (see [6, pp. 251–252 and 290–291]). Hence

$$\int_T w(t)\, x(t)\, dt = \int_T x(t)\, u'(t)\, dt = \int_T x(t)\, du(t).$$

By (13) the function u is nonconstant on T, so that the conclusion of Remark 2.3 is a consequence of formulas (5) and (14), and of Theorem 2.1. ∎

3. Superdense Unbounded Divergence of Simple Quadrature Formulas

3.1. Theorem. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied. Then* (5) *is unboundedly divergent at a point in* $C(T)$ *if and only if* $\sum_{i=1}^{\infty} |c_i| = \infty$. *Moreover, the last condition implies the unbounded divergence of* (5) *on a superdense set in* $C(T)$.

Proof. If (5) diverges unboundedly at a point x in $C(T)$, i.e., $\sup\{|x_n^*(x)| : n \in N\} = \sup\{|Q_n(x)| : n \in N\} = \infty$, then, by (3) and $|x_n^*(x)| \leqslant \|x_n^*\| \, \|x\|$, we obtain $\sum_{i=1}^{\infty} |c_i| = \sup\{\|x_n^*\| : n \in N\} = \infty$. Conversely, if $\sum_{i=1}^{\infty} |c_i| = \infty$, then from Theorem 1.3 it follows that the set $S = \{x \in C(T)$: $\sup\{|Q_n(x)| : n \in N\} = \infty\}$ is superdense in $C(T)$. Since $S \neq \varnothing$, there exist points x at which (5) is unboundedly divergent. ∎

3.2. Corollary. *Suppose that the hypotheses of Theorem* 2.1 *are satisfied and that* (5) *converges on the set* P *in Theorem* 1.2. *Then* (5) *diverges unboundedly on a superdense set in* $C(T)$.

Proof. If (5) converges on P, then Theorems 1.2 and 2.1 imply $\sum_{i=1}^{\infty} |c_i| = \infty$, so that Theorem 3.1 applies. ∎

A. I. Mitrea [5, Theorem 6.3] proved the unbounded divergence on superdense sets in $C(T)$ of quadrature formula (14) with $m_n = n$, $T = [-1, 1]$, Q_n having the general expression (2), and c_n^i having a particular form, namely c_n^i are defined by the "interpolatory formulas"

$$c_n^i = \int_T w(t)\, l_n^i(t)\, dt,$$

where l_n^i is the Lagrange interpolatory polynomial associated with the distinct nodes $t_n^1, \ldots, t_n^n$. A result of this type for simple quadrature formulas is contained in the following corollary:

3.3. Corollary. *Suppose that the hypotheses in Remark* 2.3 *are satisfied and that* (14) *converges on the set* P *in Theorem* 1.2. *Then* (14) *diverges unboundedly on a superdense set in* $C(T)$.

Proof. The argument is the same as in the proofs of Theorem 3.1 and Corollary 3.2, and it is based on Remark 2.3 instead of Theorem 2.1. ∎

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